Int. J. Heat Mass Transfer. Vol. 4, pp. 89–103. Pergamon Press 1961. Printed in Great Britain.

ON POSSIBLE METHODS OF SOLVING BOUNDARY LAYER PROBLEMS IN THE PRESENCE OF DISSOCIATION AND DIFFUSION

I. P. GINZBURG

The Zhdanov University, Leningrad, U.S.S.R.

Аннотация—В статье приводятся уравнения движения смеси газов в ламинарном и турбулентном пограничном слоях. Даётся обзор возможных методов их решения.

В частности, указаны некоторые новые классы автомодельных решений и схема приближённого решения уравнений с использованием метода Кочина-Лойцянского. Приводятся некоторые новые результаты по турбулентному пограничному слою.

NOMENCLATURE

- A, thermal equivalent of work;
- c_{pi}, heat capacity of the *i*th component of a mixture at a constant pressure;
- c_{vi}, molar heat capacity of the *i*th component at constant volume;
- D_i, diffusion coefficient of the *i*th component;
- E, activation energy;
- E^+, E^-, radiant energy full flows, incident and reflected;
- F_x, F_y, mass forces vector component (with respect to mass unit);
- f, arbitrary function of x, u and T_w;
- H, total heat content of mixture mass unit;
- h_i, mass unit heat content of the *i*th component;
- i, number of mixture components;
- J_y^i, diffusion vector component of the *i*th component;
- K_i, concentration change velocity of the *i*th component due to chemical reactions (of non-equilibrium);
- M_i molecular weight of the *i*th component;
- m, averaging coefficient;
- p, pressure;
- p_∞, pressure of an adiabatically decelerated flow;
- q_y, heat flow vector component;
- r, number of reactions when the *i*th component originates;
- T, temperature;
- T_w, wall temperature;
- u, velocity on the edge of a boundary layer;
- u_i, one mole internal energy of the *i*th component $(=u_{i0} + \int_0^T c_{v_i}\, dT)$;
- u_{i0}, internal energy of one mole at absolute zero;
- v_x, v_y, velocity vector components of a mixture as a continuous medium;
- w_α, αth reaction rate;
- a, absorption coefficient;
- δ, boundary layer thickness;
- δ^*, displacement thickness;
- δ_l, laminar sublayer thickness;
- ϵ_1, volumetric heat release ($\epsilon = 0$ for a plane problem; $\epsilon = 1$ for an axisymmetric problem);
- λ, mixture heat conductivity coefficient;
- μ, mixture viscosity coefficient;
- μ_i, viscosity of the *i*th component;
- $\nu_{i\alpha}$, stoichiometric coefficient;
- ξ_i, relative mass concentration of the *i*th mixture component $[\xi_i = (\rho_i/\rho)]$;
- ρ, mixture density;
- ρ_∞, density of an adibatically decelerated flow;
- σ, Boltzman constant;
- σ_e, conductivity;
- $\tau_{x,y}$, friction tensor component;
- χ_i, molar concentration.

1. EQUATIONS OF MOTION

THE steady equations of motion of an ideal gas mixture in the boundary layer of a wing or of axisymmetrical bodies in the presence of radiation diffusion may be written in the form:

(a) mass conservation equation:

$$\frac{\partial}{\partial x} \rho v_x r^\epsilon + \frac{\partial}{\partial y} \rho v_y r^\epsilon = 0; \tag{1.1}$$

(b) momentum equation:

$$\rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho F_x + \frac{\partial \tau_{xy}}{\partial y}; \tag{1.2}$$

$$0 = -\frac{\partial p}{\partial y} + \rho F_y; \tag{1.3}$$

(c) diffusion equation:

$$\rho\left(v_x \frac{\partial \xi_i}{\partial x} + v_y \frac{\partial \xi_i}{\partial y}\right) = -\frac{\partial}{\partial y} J_y^i + K_i; \tag{1.4}$$

(d) energy equation:

$$\rho\left(v_x \frac{\partial H}{\partial x} + v_y \frac{\partial H}{\partial y}\right) = A\rho(F_x v_x + F_y v_y) + \frac{\partial}{\partial y}(q_y + A v_x \tau_{xy}) + \frac{\partial}{\partial y}(E^+ - E^-) + \rho\epsilon_1; \tag{1.5}$$

(e) radiation transfer equation, which for a grey body (if radiant energy dispersion is to be neglected and local thermodynamic equilibrium is to be assumed) will have the form:

$$\frac{\partial E^+}{\partial y} = m\alpha(\sigma T^4 - E^+), \tag{1.6}$$

$$\frac{\partial E^-}{\partial y} = m\alpha(\sigma T^4 - E^-); \tag{1.7}$$

(f) equation of state

$$\frac{p}{\rho} = \frac{RT}{M},$$

where

$$H = \sum_i \xi_i \frac{u_i}{M_i} + \frac{p}{\rho} + A\frac{v_x^2}{2} = h + A\frac{v_x^2}{2}; \quad \frac{1}{M} = \sum_i \frac{\xi_i}{M_i}. \tag{1.8}$$

When considering motion of gas in the absence of electric and magnetic fields, mass forces may be neglected. In the case of a conducting gaseous medium, the mass forces components ρF_x and ρF_y are determined from the equations:

$$\left.\begin{aligned} \rho F_x &= -\frac{\partial}{\partial x}\frac{\mathbf{B}_x^2 + \mathbf{B}_y^2 + \mathbf{B}_z^2}{2} + \mathbf{B}_x \frac{\partial B_x}{\partial x} + \mathbf{B}_y \frac{\partial \mathbf{B}_x}{\partial y}, \\ \rho F_y &= -\frac{\partial}{\partial y}\frac{\mathbf{B}_x^2 + \mathbf{B}_y^2 + \mathbf{B}_z^2}{2} + \mathbf{B}_x \frac{\partial B_y}{\partial y} + \mathbf{B}_y \frac{\partial \mathbf{B}_y}{\partial x}, \end{aligned}\right\} \tag{1.10}$$

where $\mathbf{B}_x$, $\mathbf{B}_y$ and $\mathbf{B}_z$ are the vector components of magnetic field intensity.

The magnetic field intensity is determined by the Maxwell equations, which in the case of a boundary layer may be written in the form:

$$\left.\begin{aligned} &\frac{\partial \mathbf{B}_x}{\partial x} + \frac{\partial \mathbf{B}_y}{\partial y} = 0, \\ &\frac{1}{\sigma_e}\frac{\partial^2 \mathbf{B}_x}{\partial y^2} = -\frac{\partial}{\partial y}(v_x \mathbf{B}_y - v_y \mathbf{B}_x), \end{aligned}\right\} \tag{1.11}$$

$$\left.\begin{aligned} &\frac{1}{\sigma_e}\frac{\partial^2 \mathbf{B}_y}{\partial y^2} = -\frac{\partial}{\partial x}(v_x \mathbf{B}_y - v_y \mathbf{B}_x), \\ &\frac{1}{\sigma_e}\frac{\partial^2 \mathbf{B}_z}{\partial y^2} = \frac{\partial v_x}{\partial x}\mathbf{B}_z + \frac{\partial v_y}{\partial y}\mathbf{B}_z. \end{aligned}\right\} \tag{1.12}$$

In the case of a conducting medium, $\rho\epsilon = (\mathbf{j}^2/\sigma)$ is the Joule heat, where $\mathbf{j}$ is the current density vector ($\bar{\mathbf{j}} = \nabla \times \mathbf{B}$).

Equations (1.2) (1.4) and (1.5), the components of a friction tensor, diffusion flow vector and heat flow vector will be expressed differently in cases of laminar and turbulent boundary layers.

In the first case, if thermo-diffusion is to be neglected, then τ_{xy}, q_y and J_y^i may be determined from the following formulae:

$$\left.\begin{aligned} \tau_{xy} &= \mu\frac{\partial v_x}{\partial y}, \\ J_y^i &= \rho D_i \frac{\partial \xi_i}{\partial y}, \\ q_y &= \frac{\lambda}{c_p}\left\{\frac{\partial h}{\partial y} + \sum_i (Le_i - 1) h_i \frac{\partial \xi_i}{\partial y}\right\}, \end{aligned}\right\} \tag{1.13}$$

where

$$Le_i = \frac{\rho c_p D_i}{\lambda};\ c_p = \sum_i \xi_i \frac{\partial h_i}{\partial T_i} = \sum_i \xi_i c_{pi};$$

$$h_i = \frac{u_{i0}}{M_i} + \int_0^T c_{pi}\,\mathrm{d}T;\quad h = \Sigma\, \xi_i h_i.$$

In the case of a turbulent boundary layer, if we proceed from the semi-empirical theory of turbulence, it may be assumed [1]:

$$\left.\begin{aligned} \tau_{xy} &= \rho l^2 \left(\frac{\partial v_x}{\partial y}\right)^2, \\ J_y^i &= \rho l l_g^i \frac{\partial v_x}{\partial y}\frac{\partial \xi_i}{\partial y}, \\ q_y &= \rho l \frac{\partial v_x}{\partial y}\left[l_h \frac{\partial h}{\partial y} + \sum_i (l_d^i - l_h) h_i \frac{\partial \xi_i}{\partial y}\right], \end{aligned}\right\} \quad (1.14)$$

where l, l_d^i and l_h are mixing lengths for momentum, diffusion and heat transfer, respectively, which in boundary layer problems will be assumed to be proportional to the distance from the wall.

Moreover, considering a turbulent boundary layer, we shall assume that near the wall there is a laminar sublayer δ_l, wherein formula (1.13) is valid, the derivatives of heat content velocity and concentration undergoing discontinuity when passing through it, namely:

$$\left(\frac{\partial v_x}{\partial y}\right)_{y=\delta_{l-0}} = K_1 \left(\frac{\partial v_x}{\partial y}\right)_{y=\delta l+0},$$

$$\left(\frac{\partial h}{\partial y}\right)_{y=\delta_{l-0}} = K_2 \left(\frac{\partial h}{\partial y}\right)_{y=\delta_{l+0}},$$

$$\left(\frac{\partial \xi_i}{\partial y}\right)_{y=\delta_{l-0}} = K_3 \left(\frac{\partial \xi_i}{\partial y}\right)_{y=\delta_{l+0}}.$$

If $l_d^i = l_h = l$, then the coefficients K_1, K_2 and K_3, as can easily be shown, are connected with the relations:

$$K_3 = K_1 \frac{\mu}{\rho D_i},\quad K_2 = K_1 \frac{c_p \mu}{\lambda}.$$

Diffusion, viscosity and heat conductivity coefficients for a gas mixture depend upon temperature as well as upon concentration of certain components. At present there are no complete theoretical and experimental data on their determination for multicomponent mixtures.

There is a number of formulae for binary mixtures. For example, in Wilke's work [38] formula is suggested for the calculation of two-gases mixture viscosity:

$$\mu = \frac{\mu_1}{1 + \xi_{12}(\chi_2/\chi_1)} + \frac{\mu_2}{1 + \xi_{21}(\chi_1/\chi_2)},$$

where

$$\xi_{ij} = \frac{[1 + (M_i/M_j)^{1/4}\sqrt{(\mu_i/\mu_j)}]^2}{2\sqrt{2}\sqrt{[1 + (M_i/M_j)]}}.$$

The dependence of viscosity upon temperature is well described by the Saterland formula [2]:

$$\mu = \mu(T_\infty)\frac{T_\infty + S_{\mathrm{mix}}}{T + S_{\mathrm{mix}}}\left(\frac{T}{T_\infty}\right)^{1.5},$$

where

$$S_{\mathrm{mix}} = \frac{M_2}{2}\left(\frac{S_2}{M_2} + \frac{S_1}{M_1}\right)$$

is the Saterland constant. The dependence of heat conductivity upon temperature can be expressed through the number $Pr = (c_p\mu/\lambda)$ and the dependence $\mu(T)$.

The question of determining and calculating the viscosity coefficient, heat conductivity and diffusion for air has been discussed in detail [3, 4]. As far as the velocity of change of concentration on account of chemical reactions K_i is concerned, it may be represented in the form:

$$K_i = M_i \sum_{\alpha=1}^{r} \nu_{i\alpha} W_\alpha^i.$$

The equation of α reaction in chemical symbols is written in the form:

$$\sum_{i=1}^{q} \nu_{i\alpha} A_i = 0.$$

Since $\sum_{i=1}^{q} \mathbf{J}_i = 0$ from the determination of the diffusion vector, from equations (1.1) and (1.3) we have that $\sum_{i=1}^{q} \nu_{i\alpha} M_i = 0$.

The reaction rate W_α, according to the mass action law, will be written thus:

$$W_\alpha^i = K_\alpha^i c_1^{v_{1\alpha}} c_2^{v_{2\alpha}} \ldots\ - K_\alpha'^i c_1'^{v_{1\alpha}'} c_2'^{v_{2\alpha}'} \ldots,$$

where K_a^i and $K_a'^i$ are rate constants of a direct and inverse reaction, respectively; c_i, $c_2 \ldots$, c_1', c_2' are concentrations $[c_i = (\xi_i/M_i)\,\rho]$.

The reaction rate constants are functions of a gas mixture state. At quasi-equilibrium processes they may be considered as functions of temperature. In many cases they may be expressed according to the Arrhenius law in the form:

$$K_a^i = A_i(T)\,\mathrm{e}^{-E/RT},$$

where $A_i(T)$ is some constant or function of temperature.

The constant of a back-reaction is determined by means of the chemical equilibrium constants, K_p^{ai} and K_a^i, by the formula:

$$\frac{K_a'^i}{K_a^i} = K_p^{ai}(RT)^{-\sum_i \nu_{ia}}.$$

The question of determining reaction rate constants for dissociation of two-component mixtures and for air is discussed in [5, 6, 7, 8, etc.]. However, data on the determination of K_a^i and $K_a'^i$ in the case of dissociation and recombination are not available.

For the approximate solution of boundary layer equations it is assumed that there occurs a local chemical equilibrium. Then, concentrations, ξ_i, as functions of pressure and temperature, are determined on the basis of both the mass action law and the equations of conservation of atomic number. In this case thermodynamic functions of air have been determined and presented, for example, in tables given by Predvoditelev [9]. Equation (1.3) is excluded from consideration.

What may be the error of such an approximate solution has not yet been established.

2. TRANSFORMATION OF EQUATIONS TO THE DORODNITSIN VARIABLES

Let us introduce the values being as independent variables

$$\xi = \int_0^x f(x, u, T_w)\,\frac{p}{p_\infty}\,r^{2\epsilon}\,\mathrm{d}x,$$

$$\eta = \int_0^y \frac{\rho}{\rho_\infty}\,r^{\epsilon}\,\mathrm{d}y.$$

By introducing a new function instead of v_y:

$$V_y = \frac{\rho}{\rho_\infty}\,\frac{p_\infty}{p}\,\frac{r^{-\epsilon}}{f}\,v_y + r^{-2\epsilon}\,\frac{p_\infty}{p}\,\frac{u_x}{f}\,\frac{\partial\eta}{\partial x},$$

and turning to dimensionless variables $u = u_{\max}\,\bar{u}$

$$v_x = \bar{v}_x u_{\max}, \quad V_y = \bar{V}_y\,\frac{\nu_\infty}{L u_{\max}}\,u_{\max},$$

$$\xi = \bar{\xi}\,\frac{u_{\max} L^2}{\nu_\infty}, \quad \eta = \bar{\eta} L, \quad H = H_\infty(1+g),$$

$$h_i = H_\infty(1+g_i), \quad Z_i = \frac{\xi_i}{\xi_{i\delta}}$$

where

$$u_{\max}^2 = \frac{2H_\infty}{A};$$

L is the characteristic length having the order of thickness of a boundary layer; after certain transformations the equations determining the laminar motion of a gas mixture (ignoring radiation) may be written in the form (bars omitted in future):

$$\frac{\partial v_x}{\partial \xi} + \frac{\partial V_y}{\partial \eta} = 0, \tag{2.1}$$

$$v_x\,\frac{\partial v_x}{\partial \xi} + V_y\,\frac{\partial v_x}{\partial \eta} = \frac{\rho_0}{\rho}\,u\,\frac{\mathrm{d}u}{\mathrm{d}\xi} + \frac{\partial}{\partial \eta}\left(K\,\frac{\partial v_x}{\partial \eta}\right), \tag{2.2}$$

$$v_x\,\frac{\partial Z_i}{\partial \xi} + V_y\,\frac{\partial Z_i}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\frac{K}{Sc_i}\,\frac{\partial Z_i}{\partial \eta}\right) + \bar{W}_i \tag{2.3}$$

$$\left.\begin{aligned} v_x\,\frac{\partial g}{\partial \xi} + V_y\,\frac{\partial g}{\partial \eta} = \frac{\mu_\infty}{\rho_\infty}\,\frac{\partial}{\partial \eta}\,K\,\Bigg[&\frac{1}{Pr}\,\frac{\partial g}{\partial \eta} \\ &+ \left(1 - \frac{1}{Pr}\right)\frac{\partial}{\partial \eta}\,\frac{v_x^2}{2} \\ &+ \sum_i \frac{Le_i}{Pr}\left(1 - \frac{1}{Le_i}\right)\xi_{i\delta}\,(1+\tilde{g}_i)\,\frac{\partial Z_i}{\partial \eta}\Bigg], \end{aligned}\right\} \tag{2.4}$$

where

$$K = \frac{\mu\rho p_\infty}{\mu_\infty \rho_\infty p}\,\frac{1}{f} = \frac{N}{f}, \quad Pr = \frac{c_p \mu}{\lambda},$$

$$\bar{W}_i = K_i\,\frac{p_\infty}{p\rho}\,\frac{r^{-2\epsilon}}{f}\,\frac{L^2}{\nu_\infty \xi_{i\delta}}, \quad \frac{1}{Sc_i} = \frac{Le_i}{Pr}.$$

Let us represent boundary conditions for system, equations (2.1) to (2.4) as follows:

$$\left.\begin{array}{l}\eta = 0, \quad v_x = 0, \quad V_y = V_0, \quad Z_i = Z_{wi}, \\ g = \dfrac{H_w}{H_\infty} - 1 = g_w, \quad \tilde{g}_i = \dfrac{h_{iw}}{H_\infty} - 1 \\ \eta \to \infty \; v_x = u, \quad Z_i = 1, \quad g = 0.\end{array}\right\} \quad (2.5)$$

In the absence of volumetric heat release and mass forces, provided that H_∞ and $\xi_{i\delta}$ are assumed to be constant, integral relations in the Dorodnitsin variables will be written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\delta^{**} + \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}\xi}(2\delta^{**} + \delta^*) = \frac{K_w}{u}\nu_\infty\left(\frac{\partial}{\partial\eta}\frac{v_x}{u}\right)_{\eta=0} + V_w, \quad (2.6)$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(\delta_s^* u) + uV_w\left(\frac{\xi_{iw}}{\xi_{i\delta}} - 1\right) = \frac{\rho_w D_{iw}}{\mu_w} K_w \nu_\infty\left(\frac{\partial Z_i}{\partial\eta}\right)_{\eta=0} + \int_0^\infty \frac{p_\infty}{p f}\frac{W_i}{\rho r^{2\epsilon}}\,\mathrm{d}\eta, \quad (2.7)$$

$$\left.\begin{array}{l}\dfrac{\mathrm{d}}{\mathrm{d}\xi}(u\Delta^{**}) + uV_w\left(\dfrac{H_w}{H_\infty} - 1\right) \\ = \dfrac{K_w}{Pr_w}\nu_\infty\left\{\left(\dfrac{\partial}{\partial\eta}\dfrac{H}{H_\infty}\right)_{\eta=0}\right. \\ \left. + \sum_i \dfrac{\rho_w D_{iw}}{\mu_w} Pr_{iw}\left(1 - \dfrac{1}{Le_{iw}}\right)\dfrac{h_{iw}}{H_\infty}\dfrac{\partial\xi_i}{\partial\eta}\bigg|_{\eta=0}\right\}.\end{array}\right\} \quad (2.8)$$

Here

$$\left.\begin{array}{l}V_w = \dfrac{\rho_w v_w}{\rho_\infty u}\dfrac{p_\infty}{p}\dfrac{r^{-\epsilon}}{f}, \\ \delta^{**} = \displaystyle\int_0^{\delta_1}\frac{u_x}{u}\left(1 - \frac{v_x}{u}\right)\mathrm{d}\eta, \\ \delta^* = \displaystyle\int_0^{\delta_1}\left(\frac{\rho_0}{\rho} - \frac{v_x}{u}\right)\mathrm{d}\eta, \\ \delta_s^* = \displaystyle\int_0^{\delta_3}\frac{v_x}{u}(1 - Z_i)\,\mathrm{d}\eta, \\ \Delta^{**} = \displaystyle\int_0^{\delta_2}\frac{v_x}{u}\left(1 - \frac{H}{H_\infty}\right)\mathrm{d}\eta\end{array}\right\} \quad (2.9)$$

where δ_1, δ_2 and δ_3 respectively denote thicknesses of dynamic, thermal and diffusion boundary layers in the Dorodnitsin variables. The thicknesses mentioned are determined from the conditions:

$$\frac{\partial v_x}{\partial\eta}\bigg|_{\eta=\delta_1} = 0, \quad \frac{\partial Z_i}{\partial\eta}\bigg|_{\eta=\delta_3} = 0, \quad \frac{\partial H}{\partial\eta}\bigg|_{\eta=\delta_2} = 0.$$

[It is assumed in equation (2.8) that $(q_y + Av_x\tau_{xy})_{\eta=\delta_2} = 0$.] In the case of an asymptotic boundary layer in equation (2.3) it must be assumed that $\delta_1 = \delta_2 = \delta_3 = \infty$. In the case of equilibrium dissociation with the assumption that $(T/M) = ah + b$ the thickness of displacement δ^* is represented in the form:

$$\delta^* = \frac{1}{1 - A(u^2/2H_\infty) + (b/aH_\infty)} \times\left\{\frac{b}{aH_\infty}\delta_1^\cdot + \Delta^* + A\frac{u^2}{2H_\infty}\delta^*\right\}$$

where

$$\delta_1^\cdot = \int_0^{\delta_1}\left(1 - \frac{v_x}{u}\right)\mathrm{d}\eta, \quad \Delta^* = \int_0^{\delta_1}\left(\frac{H}{H_\infty} - \frac{v_x}{u}\right)\mathrm{d}\eta.$$

3. POSSIBLE SIMILARITY SOLUTIONS OF LAMINAR BOUNDARY LAYER EQUATIONS

Let us find out at what values of $u(\xi)$, $V_0(\xi)$, $g_w(\xi)$ and $Z_{iw}(\xi)$ solutions of equations (2.1) to (2.4) dependent on one variable $\tau = \eta x(\xi)$ are possible.

Assuming

$$\frac{v_x}{u} = \frac{\partial F(\tau, \xi)}{\partial\tau}$$

$$g = g_w\varphi(\tau, \xi),$$

$$Z_i = Z_i(\tau, \xi),$$

from equation (2.1) we shall have:

$$V_y = V_0 - \frac{u}{x}\frac{\partial F}{\partial\xi} - \frac{ux'}{x^2}\tau\frac{\partial F}{\partial\tau} + F\left(\frac{ux'}{x'} - \frac{u'}{u}\right) \quad (3.1)$$

and equations (2.2) to (2.4) will be reduced to the form:

$$\left.\begin{aligned}&\left(\frac{\partial F}{\partial\tau}\frac{\partial^2 F}{\partial\tau\partial\xi}-\frac{\partial F}{\partial\xi}\frac{\partial^2 F}{\partial\tau^2}\right)\frac{c_6}{(\mathrm{d}/\mathrm{d}\xi)\ln(x/u)}\\&+\left[\frac{V_0 f}{x}+c_6 F\right]\frac{\partial^2 F}{\partial\tau^2}\\&=\frac{\mathrm{d}u/\mathrm{d}\xi}{u}\frac{c_6}{(\mathrm{d}/\mathrm{d}\xi)\ln(x/u)}\\&\left[\frac{\rho_0}{\rho}-\left(\frac{\partial F}{\partial\tau}\right)^2\right]+\frac{\partial}{\partial\tau}N\frac{\partial F}{\partial\tau};\end{aligned}\right\}\quad(3.2)$$

$$\left.\begin{aligned}&\frac{c_6}{(\mathrm{d}/\mathrm{d}\xi)\ln(x/u)}\left\{\frac{\mathrm{d}\ln g_w}{\mathrm{d}\xi}\frac{\partial F}{\partial\tau}\varphi+\frac{\partial F}{\partial\tau}\frac{\partial\varphi}{\partial\xi}\right.\\&\left.-\frac{\partial F}{\partial\xi}\frac{\partial\varphi}{\partial\tau}\right\}+\left[\frac{V_0 f}{x}+c_6 F\right]\frac{\partial\varphi}{\partial\tau}\\&=\frac{\partial}{\partial\tau}N\left[\frac{1}{Pr}\frac{\partial\varphi}{\partial\tau}+\left(1-\frac{1}{Pr}\right)\frac{u^2}{g_w}\frac{\partial}{\partial\tau}\right.\\&\times\left(\frac{\partial F}{\partial\tau}\right)^2+\sum_i\frac{Le_i}{Pr}\left(1-\frac{1}{Le_i}\right)\\&\left.\times\frac{\xi_i\delta}{g_w}(1+\tilde{g}_i)\frac{\partial Z_i}{\partial\tau}\right];\end{aligned}\right\}\quad(3.3)$$

$$\left.\begin{aligned}&\frac{c_6}{(\mathrm{d}/\mathrm{d}\xi)\ln(x/u)}\left\{\frac{\mathrm{d}}{\mathrm{d}\xi}\ln\xi_{i\delta}Z_i\frac{\partial F}{\partial\tau}+\frac{\partial F}{\partial\tau}\frac{\partial Z_i}{\partial\xi}\right.\\&\left.-\frac{\partial Z_i}{\partial\tau}\frac{\partial F}{\partial\xi}\right\}+\left[\frac{V_0 f}{x}+c_6 F\right]\frac{\partial Z_i}{\partial\tau}\\&=\frac{\partial}{\partial\tau}\left(N\frac{Le_i}{Pr}\frac{\partial Z_i}{\partial\tau}\right)+\tilde{W}_i,\end{aligned}\right\}\quad(3.4)$$

where

$$\tilde{W}_i=W_i\frac{f}{x^2};\quad c_6=\frac{fu}{x^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\ln\frac{x}{u}.$$

As is seen from the system of equations given F, φ and Z_i will be functions of τ if the following conditions are fulfilled:

$$N=N_1(\xi)N_2(\tau),$$

$$\left.\begin{aligned}&\frac{V_0 f}{N_1 x}=c_3=\text{const},\\&\frac{c_6}{N_1}=-c_1=\text{const},\end{aligned}\right\}(3.5)$$

$$\left.\begin{aligned}&w_2(\xi)\frac{f(\mathrm{d}u/\mathrm{d}\xi)}{N_1x^2}=\text{const},\\&w_4(\xi)\frac{f(\mathrm{d}u/\mathrm{d}\xi)}{N_1x^2}=\text{const},\end{aligned}\right\}(3.6)$$

where it is assumed that

$$\frac{\rho_0}{\rho}-\left(\frac{\partial F}{\partial\tau}\right)^2=w_1(\tau)w_2(\xi)+w_3(\tau)w_4(\xi),$$

$$\left(1-\frac{1}{Pr}\right)\frac{u^2}{g_w},\quad Pr,$$

$$\left(1-\frac{1}{Le_i}\right)\frac{(1+\tilde{g}_i)\xi_{i\delta}}{g_w}\frac{Le_i}{Pr},\quad\tilde{W}_i$$

are functions of τ or constants;

$$\left.\begin{aligned}&\frac{c_6}{N_1(\mathrm{d}/\mathrm{d}\xi)\ln(x/u)}\frac{\mathrm{d}\ln g_w}{\mathrm{d}\xi}\\&\text{and}\\&\frac{c_6}{N_1(\mathrm{d}/\mathrm{d}\xi)\ln(x/u)}\frac{\mathrm{d}\ln\xi_{i\delta}}{\mathrm{d}\xi}\end{aligned}\right\}\quad(3.7)$$

are constant values. If the assumption that $(T/M)=ah+b$ is valid, then:

$$\left.\begin{aligned}&w_1(\tau)=\varphi(\tau),\\&w_2(\xi)=\frac{g_w}{1-u^2+(b/aH_\infty)},\\&w_3(\tau)=1+\frac{b}{aH_\infty}\left[1-\left(\frac{\partial F}{\partial\tau}\right)^2\right]-\left(\frac{\partial F}{\partial\tau}\right)^2,\\&w_4(\xi)=\frac{1}{1-u^2+(b/aH_\infty)}.\end{aligned}\right\}\quad(3.8)$$

In the case of a plate or a cone, the conditions of equation (3.6) are always fulfilled and the remainder with the assumption that $f=N_1$ give the following equations for determining $x(\xi)$, $V_0(\xi)$, $g_w(\xi)$ and $\xi_{i\delta}(\xi)$:

$$\left.\begin{aligned}&x(\xi)=\frac{1}{\sqrt{\{2[(c_1/u)\,\xi+c_2]\}}},\quad V_0=c_3x,\\&\xi_{i\delta}=c_8\left(\xi+\frac{c_2u}{c_1}\right)^{c_6/c_7},\\&g_w=c_5\left(\xi+\frac{c_2u}{c_1}\right)^{c_4/c_1},\end{aligned}\right\}\quad(3.9)$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ are arbitrary constants. And

$$N_2,\quad Pr,\quad \frac{Le_i}{Pr},\quad \left(1 - \frac{1}{Le_i}\right)(1 + \tilde{g}_i)\frac{\xi_{i\delta}}{g_w}$$

have to be functions of τ or constants.

Similar solutions of equations (3.2) to (3.4) are presented, for example, in the case of a plate for some particular cases [10–13]. Ordinary equations obtained in this case are simply solved when Pr and Le_i are constant, $N_2(\tau) =$ const, $g_w =$ const, $\xi_{i\delta} =$ const, $\tilde{W}_i = 0$, and injection is absent. In the presence of injection and variable numbers Pr and Le_i the equations are solved numerically. It will be noted that the solution of the ordinary equations obtained in the presence of chemical reactions, i.e. at $\tilde{W}_i \neq 0$, has not yet been actually considered (see, however, [12] and [14]). The case with variables g_w, $\xi_{i\delta}$ has not been considered either.

In the case of a wing or axisymmetrical body the conditions that $[1 - (1/Pr)]\, u^2/g_w$ is the function of τ or the constant (since u is the function of ξ) may be fulfilled only when:

(a) $Pr = 1$

(b) $\dfrac{u^2}{g_w} =$ const,

and Pr is the function of τ or the constant.

In the case of hypersonic velocities, when a boundary layer of thin bodies is considered, it may be assumed that $u^2 \approx 1$, then the condition $u^2/g_w =$ const will be fulfilled at $g_w =$ const. When a boundary layer near a stagnation point is considered, u^2 may be assumed approximately equal to zero. The conditions of equation (3.6) are fulfilled not at each dependence ρ_0/ρ upon T. In particular, for their fulfillment it is necessary that $w_2(\xi)/w_4(\xi) =$ const, which with the assumption that the equation $(T/M) = ah + b$ is valid leads to the condition $g_w =$ const.

In this case equations (3.5) and (3.6) lead to the following equations for determining $x(\xi)$, $u(\xi)$ and $V_0(\xi)$ where there are similar solutions:

$$f V_0 = c_3 N_1 x,\quad \frac{fu}{N_1 x^2}\left(\frac{x'}{x} - \frac{u'}{u}\right) = -c_1$$

$$\frac{f\,(\mathrm{d}u/\mathrm{d}\xi)}{N_1 x^2[1 - u^2 + (b/aH_\infty)]} = \beta. \tag{3.10}$$

From the latter equations we get:

$$x = \sqrt{\left(\frac{f\,(\mathrm{d}u/\mathrm{d}\xi)}{\beta N_1[1 - u^2 + (b/aH_\infty)]}\right)} = \sqrt{\left(\frac{(p_\infty/p)\, r^{-2\epsilon}\,(\mathrm{d}u/\mathrm{d}x)}{\beta N_1[1 - u^2 + (b/aH_\infty)]}\right)} \tag{3.11}$$

and for $u(\xi)$, the following equation:

$$\frac{f}{N_1}\,\frac{\mathrm{d}u/\mathrm{d}\xi}{\beta[1 - u^2 + (b/aH_\infty)]} = D^2\,\frac{u^{-(1/m)+2}}{[1 - u^2 + (b/aH_\infty)]^{-1/2m}}$$

or

$$\frac{(p_\infty/p)\, r^{-2\epsilon}\,(\mathrm{d}u/\mathrm{d}x)}{N_1\beta[1 - u^2 + (b/aH_\infty)]} = D^2\,\frac{u^{-(1/m)+2}}{[1 - u^2 + (b/aH_\infty)]^{-1/2m}},$$

where

$$\frac{1}{2m} = \frac{c_1}{\beta[1 + (b/aH_\infty)]}.$$

The term entering equation (3.2)

$$\frac{1}{u}\,\frac{\mathrm{d}u}{\mathrm{d}\xi}\,\frac{c_6}{(\mathrm{d}/\mathrm{d}\xi)\ln(x/u)}\left[\frac{\rho_0}{\rho} - \left(\frac{\partial F}{\partial \tau}\right)^2\right]$$

$$= \beta\left[1 + g_w\varphi - \left(\frac{\partial F}{\partial \tau}\right)^2\left(1 + \frac{b}{aH_\infty}\right) + \frac{b}{aH_\infty}\right].$$

The form of the function $u(x)$ depends on parameters β, m and D, and that of $u(\xi)$ on the form of the function $f(\xi)$ as well.

In particular, assuming that

$$f = N_1\left(1 - u^2 + \frac{b}{aH_\infty}\right)^{1+(1/2m)},$$

we shall get $u(\xi)$ in the form:

$$u = c_{11}(\xi + c_{12})^n \tag{3.13}$$

if $m \neq 1$, and

$$u = c_{11}\,\mathrm{e}^{D^2\beta\xi} \tag{3.14}$$

if $m = 1$, where $n = [m/(1 - m)]$,

$$x(\xi) = D\,\frac{u^{-(1/2m)+1}}{\{\sqrt{[1 - u^2 + (b/aH_\infty)]}\}^{-1/2m}},$$

c_{11}, c_{12} are arbitrary constants.

If we assume $f = uN_1$ we shall obtain

$$x = u\sqrt{\left(\frac{1}{2(c_1\xi + D)}\right)} \tag{3.15}$$

and

$$u = c\,\frac{[\xi + (D/c_1)]^m \sqrt{[1 + (b/aH_\infty)]}}{\sqrt{\{1 + c^2[\xi + (D/c_1)]^{2m}\}}}.$$

The variables ξ and τ turning into the Mangler variables.

The numerical solutions of equations (3.2) and (3.3) under boundary conditions:

$$F(0) = F'(0) = 0; \quad F'(\infty) = 1;$$

$$\varphi(0) = 1; \qquad \varphi(\infty) = 0.$$

at $b = 0$, $N_1 = 1$, $Pr = 1$, $Le_i = 1$ and $m \neq 1$ over a wide range of β (up to $\beta = 2$) were obtained by a number of investigators [15–17]. Moreover, Cohen and Reshotko [17] obtained a numerical solution of the mentioned equations at $Pr = 0{\cdot}7$ and $(2/g_w) = 1{\cdot}2$.

Similar solutions may also be used in studying a motion near the critical point where $u \approx x(\mathrm{d}u/\mathrm{d}x)_0$. It is convenient to introduce here the Mangler variables. Numerical solutions of equations for this case are presented by Reshotko and Cohen [18] for $Le_i = 1$, $N_1 = 1$ and by Li and Nagamatsi [16] and by Lees [19] for $Pr = 1$ and $Pr = 0{\cdot}7$. In the neighbourhood of the critical point at $Le_i \neq 1$ for final reaction rates for a catalytic and non-catalytic surface, the numerical solution of equations (3.2) to (3.4) was obtained by Fay, Riddell and Kemp [20, 21]. The solution of these equations for injection and also for another change law of $u(\xi)$ has not been discussed in detail. Similar solutions for chemical reaction and surface melting also demand a more detailed analysis.

4. APPROXIMATE METHODS FOR SOLVING LAMINAR BOUNDARY LAYER EQUATIONS

As was mentioned above, similar solutions may occur both with definite velocity distribution of external flow along the edge of the boundary layer and for certain ranges of the numbers Pr, Le_i, g_w, etc. In the general case of arbitrary pressure distribution along a profile and with arbitrary boundary conditions the problem has to be solved approximately.

In the boundary-layer theory the following arbitrary methods are applied:

(1) A partial application of local similarity in which the pre-history of a flow is taken into consideration only as the dependence of coefficients on the variable ξ.

(2) Methods of integral relations.

(3) Methods involving expansion in series in which the coefficients of each variable are obtained by solving ordinary differential equations, and the expansion variables depend on the conditions outside the boundary layer.

(4) Methods of successive approximation.

(5) Difference methods for solving laminar boundary-layer equations.

(6) Methods of small disturbances.

We shall now consider on these methods:

(1) *Local similar solutions*

It is the essence of this method that the problem is reduced to solving systems (3.2) and (3.4) with local values β, g_w, (u^2/H_∞) and N determined at each point of ξ.

Such a method is applicable if it is known beforehand that the external flow and gas properties on a body depend little on ξ, i.e. when derivatives with respect to ξ are small compared with those with respect to η.

(2) *Method of integral relations*

The application of this method to solving the problem of gas motion in a laminar boundary layer is different both for a finite thickness layer and a symptotic one. For the layer of finite thickness it is assumed that the profile of velocities, heat contents and concentration may be represented as polynomials of ratios $(1/\delta_i)$, where δ_i is the corresponding thickness, the coefficients of which are determined from the conditions on the wall and on the edge of a boundary layer. From the integral relations we get ordinary differential equations for determining boundary layer thicknesses. The wall conditions are obtained from the differential equations, assuming their validity on the wall, their number being able to increase by differentiating the equations. For a thermo-isolated profile the method in question was used in a number of works [22–24]. The calculations become rather complicated under more general wall conditions.

For the asymptotic boundary layer, the integral relation method is used, with corresponding profiles taken from a similar solution. In the foreign literature such a conception was first used by Thwaites [25], who considered a definite relation in which tangential wall stress, local pressure gradient and the ratio of displacement thickness to the impulse loss thickness are interrelated. The Thwaites method was worked out by Rott and Crabtrec [26] who considered the case for $Pr = 1$ and a thermo-isolated surface, and also by Cohen and Reshotko [17] for $Pr = 1$, g_w is constant. In Cohen and Reshotko's version the enthalpy gradient on the wall, $g_w\varphi'(0)$, is assumed to be, as a result of similar solution, the known function β of the parameter connected by the definite relation with the pressure gradient and impulse loss thickness.

Later on this method was worked out by Hayes [27] who, in contrast to Thwaites, introduced the boundary layer thickness as the function $G(x)$ instead of the impulse loss thickness to separate the variables.

Methods utilizing local similar solutions are similar to the Kochin–Loitsyansky method [28] worked out for an incompressible liquid and generalized for a compressible one.

We shall now show how it is possible to solve a problem on a gas mixture motion in a laminar boundary layer by the method mentioned above.

If we assume that $f_1 = N_w$ and introduce the values $\delta^{**} = \delta^{**}x$, $\delta_g^* = x\delta_g^*$, $\Delta^{**} = -(x/g_w)\Delta^{**}$ and substitute $(v_x/u) = (\partial F/\partial\tau)$ etc., the integral equations (2.6) to (2.8) may be written as follows:

$$\left.\begin{aligned} & u\delta^{**}\frac{d\delta^{**}}{d\xi} + \beta(\delta^{**})^2\left\{\left(1+\frac{b}{aH_\infty}\right)\frac{\delta_1^*}{\delta^{**}}\right. \\ & \left. + g_w\frac{\Delta^*}{\delta^{**}} + \left[2\left(1+\frac{b}{aH_\infty}\right) - u^2\right]\right\} \\ & = \delta^{**}F''_{\tau\tau}(0,\xi) + \beta_1; \end{aligned}\right\} \quad (4.1)$$

$$u\delta_g^*\left(\frac{d\delta_g^*}{d\xi} + \delta_g^*\frac{d\ln u}{d\xi}\right)$$

$$= \beta_1\delta_g^*(1 - Z_{iw}) + \left(\frac{Le_i}{Pr}\right)_w \delta_g^* Z'_{i\tau}(0,\xi); \quad (4.2)$$

$$\left.\begin{aligned} & u\Delta^{**}\left(\frac{d\Delta^{**}}{d\xi} + \Delta^{**}\frac{d\ln u}{d\xi}\right) - \beta_1 g_w^2\Delta^{**} \\ & = \frac{g_w}{Pr_w}\left\{ g_w\varphi'_\tau(0,\xi)\Delta^{**}\right. \\ & \left. + \sum_i (Le_i - 1)\frac{h_{iw}\xi_{i\delta}}{H_\infty}\Delta^{**}z'_\tau(0,\xi)\right\} \end{aligned}\right\} \quad (4.3)$$

where

$$\beta = \frac{\delta^{**2}}{\delta^{**2}}\frac{du/d\xi}{1 - u^2 + (b/aH_\infty)},$$

$$\beta_1 = \frac{V_0\delta^{**}}{\delta^{**}} = \frac{V_0}{x}. \quad (4.4)$$

For the approximate solution of the problem of gas motion in a boundary layer of the wing, when there are no similar solutions, let us assume that:

$$\frac{v_x}{u} = \frac{\partial F}{\partial\tau}, \quad \varphi = \varphi(\tau_3), \quad Z_i = Z_i(\tau_2),$$

where

$$\tau_1 = \frac{\eta}{\delta^{**}}\delta^{**}, \quad \tau_2 = \frac{\eta}{\delta_g^*}\delta_g^*, \quad \tau_3 = \frac{\eta}{\Delta^{**}}, \Delta_g^*$$

and the form of the functions $F(\tau_1)$, $\varphi(\tau_3)$ and $Z_i(\tau_2)$ are the same as in the case of similar solutions. In this case we shall consider the parameters β, β_1, g_w, Z_{iw} as functions of ξ; β, β_1 being expressed through δ^{**} and $u(\xi)$ from equations (4.4) where:

$$\delta^{**} = \int^\infty \frac{\partial F}{\partial\tau}\left(1 - \frac{\partial F}{\partial\tau}\right) d\tau,$$

$$\delta_g^* = \int_0^\infty \frac{\partial F}{\partial\tau}(1 - Z_i)\, d\tau,$$

and so on.

The values entering the integral equations $\delta^{**}F_{\tau\tau}(0,\xi)$, δ_1^*/δ^{**} as functions of β and β_1; Δ^*/δ^{**}, $\varphi'_\tau(0,\xi)\Delta^{**}$ as functions of β, β_1, g_w, Le_i and Pr; δ_g^*/δ^{**}, $\delta_g^* Z'_{i\tau}(0,\xi)$ as functions of β, β_1;

Z_{iw}, Le_i/Pr and $\bar{\Delta}^{**}/\delta^{**}$ as a function of β, β_1, g_w, Z_{iw}, Le_i and Pr are determined from similar solutions.

In solving the problem, δ^{**} is first determined as a function of ξ at given $u(\xi)$ and V_0 from equation (4.1) then $\beta(\xi)$ and $\beta_1(\xi)$, and lastly δ_g^* are determined. Friction, heat transfer and the diffusion flow are determined from the relations:

$$\tau_w = \mu_w \frac{\rho_w}{\rho_\infty} r^\epsilon u F''_{\tau\tau}(0, \xi) \frac{\delta^{**}}{\delta^{**}},$$

$$J_w^i = -\frac{\rho_w^2}{\rho_\infty} D_{iw} r^\epsilon \xi_{i\delta} Z'_{i\tau}(0, \xi) \frac{\delta_g^*}{\delta_g^*},$$

$$g_w = \mu_w \frac{\rho_w}{\rho_\infty} r^\epsilon \left\{ \frac{g_w}{H_\infty} \frac{\varphi'_\tau(0, \xi)}{\Delta^{**}} \Delta^{**} \frac{1}{Pr_w} + \sum_i \frac{Le_i - 1}{Pr_w} h_{iw} \xi_{i\delta} \frac{Z'_{i\tau}(0, \xi)\delta_g^*}{\delta_g^*} \right\}.$$

Equation (4.1) for the incompressible liquid was solved by Kochin and Loitsyansky. The solution of the latter equations for $Le_i = 1$ at $g_w = 0$ has been obtained [29].

The works devoted to the consideration of more general cases, as far as we know, are not available.

(3) *Expansion methods*

For simplicity we shall consider thermodynamic equilibrium at $Le_i = 1$. To apply the expansion method it is necessary to express all the parameters characterizing gas properties, (e.g. the density ratio ρ_0/ρ) analytically.

The velocity outside a boundary layer is represented as follows

$$u = u_0\xi^{1/4} + u_1\xi^{3/4} + u_2\xi^{5/4} \ldots = \sum_{K=0}^{\infty} u_K \xi^{(2K+1)/4}, \tag{4.5}$$

where u_K is a constant, and is determined from the non-viscous velocity distribution.

The unknown functions of F and g^* are also given as power series of ξ.

$$F = F_s + \sum_{K=1}^{\infty} F_K \xi^{K/2}, \quad g = g_s + \sum_{K=1}^{\infty} g_K \xi^{K/2}, \tag{4.6}$$

where F_K and g_K are considered to be dependent only on τ.

The first term in equation (4.5) is a non-viscous flow in the neighbourhood of the axi-symmetrical critical point. Such a flow has been examined before, so that F_s and g_s are known. Substituting equation (9.6) into the equations and comparing coefficients at equal powers of ξ, we get ordinary differential equations for determining F_K and g_K. In the general case this method leads to very laborious calculations. Nevertheless, it can be simple enough if the non-viscous velocity distribution is represented by two or three terms of series of equation (4.5).

This method was used by Görtler [30] for a two-dimensional incompressible boundary layer without heat transfer; and by Sparrow [31] for the same problem with heat transfer. It can also be generalized for the case $Le_i \sim 1$, as well as for cases where diffusion is taken into account.

(4) *Methods of successive approximations*

This method applied to an incompressible liquid is given in Shvets' work [32]. The essence of this method is as follows. In the system, equations (1.2) to (1.4), for a first approximation inertial terms are neglected, and the solution of a simplified system with corresponding boundary conditions is found. The next approximation is obtained by substituting the first one into the inertial terms, and solving the system obtained for zero boundary conditions. The sum of this solution, and the first approximation gives the second approximation and so on. The thicknesses of dynamic, thermal and diffusion boundary layers are determined from the conditions:

$$\left(\frac{\partial v_x}{\partial y}\right)_{y=\delta_1} = 0, \quad \left(\frac{\partial h}{\partial y}\right)_{y=\delta_2} = 0, \quad \left(\frac{\partial \xi_i}{\partial y}\right)_{y=\delta_3} = 0.$$

Applying this method to a compressible liquid, it is convenient to introduce the variable $\bar{\eta} = \int_0^y (\mu/\mu_\infty)\,dy$, instead of the variable η.

Using this method Galanova [33] solved the problem for a laminar boundary layer of a plate in the presence of dissociation. The solution of the problem of a laminar boundary layer both of a plate and a wing in the presence of injection predetermined by the arbitrary law is given in work [34]. A good agreement of the results obtained by this method with the available exact solutions should be noted. The laborious nature of the calculations is a drawback of this method.

(5) *Difference methods*

Methods for the solution of laminar boundary layer equations employing assumptions (similarity, predetermined type of a velocity profile and enthalpy, velocity expansion outside the boundary layer by ξ with coefficients independent of ξ, etc.) which facilitated the problem of reducing a non-linear system in partial derivatives to one or several non-linear ordinary equations, were considered in previous sections. But even with such simplifications one has to resort to a numerical solution of the equations obtained. It is therefore of interest to consider methods for directly solving the partial differential equations without applying assumptions of similarity, etc.

For simplicity let us consider the case of thermodynamic equilibrium at $Le_i = 1$ [35].

Turning to Crocco's variables:

$$\bar{v}_x = \frac{v_x}{u}, \quad \xi = \int_0^x r^{2\epsilon} \rho_0 u \, dx$$

and introducing the unknown functions:

$$\bar{\tau} = \frac{\mu\,(\partial v_x/\partial y)\,\sqrt{2\xi}}{r^{\epsilon} \rho_0 \mu_0 u^2}, \quad H' = \frac{H - H_w}{H_\infty - H_w},$$

we shall write equations for determining $\bar{\tau}$ and H' as follows:

$$\left.\begin{aligned} &A_1 \frac{\partial^2 \tau}{\partial \bar{v}_x^2} + B_1 \frac{\partial \bar{\tau}}{\partial \xi} + C_1 \frac{\partial H'}{\partial \xi} + D_1 \frac{\partial \tau}{\partial \bar{v}_x} \\ &\quad + E_1 \frac{\partial H'}{\partial \bar{v}_x} + F_1 = 0, \\ &A_2 \frac{\partial^2 H'}{\partial \bar{v}_x^2} + B_2 \frac{\partial H'}{\partial \xi} + C_2 \frac{\partial \bar{\tau}}{\partial \xi} + D_2 \frac{\partial H'}{\partial \bar{v}_x} \\ &\quad + E_2 \frac{\partial \bar{\tau}}{\partial \bar{v}_x} + G_2 \left(\frac{\partial \bar{\tau}}{\partial \bar{v}_x}\right)\left(\frac{\partial H'}{\partial \bar{v}_x}\right) + F_2 = 0, \end{aligned}\right\} \quad (4.7)$$

where the coefficients A_1, B_1, C_1, D_1, E_1, F_1, A_2, B_2, C_2, D_2, E_2, F_2 and G_2 are complex functions of all the initial values and also of ξ, $\bar{v}_x$, $\bar{\tau}$, h, Pr, $du/d\xi$ and $K = (c_p/c_v)$.

Boundary conditions will take the form* at

$$\bar{v}_x = 0, \quad \frac{\partial \tau}{\partial \bar{v}_x} = f_1\left(\xi, u, \frac{p_0}{p_s}, \frac{du}{d\xi}, \frac{h_0}{h_s}\right)$$

$$\bar{v}_x = 1, \quad \tau = 0, \quad H' = 0. \qquad (4.8)$$

It can be shown that the system presented is of a parabolic type, and the characteristics are the lines ξ = const. It makes it possible to apply approximate methods using the connexion between the values of the function and those of its derivatives at three neighbouring points to the system presented. As a result the problem is reduced either to the solution of n algebraic equations with n unknowns, or to the system of the ordinary differential equations on each line ξ = const. There arise difficulties in obtaining data on the singular line.

The difference approximation of derivatives has been represented by ξ as well as by $\bar{v}_x$ in the form [35]:

$$\left.\begin{aligned} &\frac{\partial H'}{\partial \xi} = \frac{H_i' - H_b'}{\Delta \xi}, \\ &\frac{\partial H'}{\partial \bar{v}_x} = \frac{1}{4\Delta \bar{v}_x}(H_a' - H_c' + H_d' - H_f'), \\ &\frac{\partial^2 H'}{\partial \bar{v}_x^2} = \frac{1}{2\Delta \bar{v}_x^2}(H_a' - 2H_b' + H_c' + H_d' \\ &\qquad\qquad - 2H_e' + H_f'), \\ &\left(\frac{\partial H'}{\partial \bar{v}_x}\right)\left(\frac{\partial \bar{\tau}}{\partial \bar{v}_x}\right) \\ &\quad = \frac{1}{8\Delta \bar{v}_x^2}[(H_a' - H_c')(\bar{\tau}_d - \bar{\tau}_f) \\ &\qquad + (H_d' - H_f')(\bar{\tau}_a - \bar{\tau}_c)] \end{aligned}\right\} \quad (4.9)$$

where symbols d, e, f, a, b, c refer to the series of three points chosen on the successive lines ξ = const.

On each line ξ = const the system is reduced to $2(i-1)$ linear equations with $2(i-1)$ unknowns, H_i' and τ_i, of the form:

$$A_{1i}H_{i+1}' + B_{1i}H_i' + c_{1i}H_{i+1}' + D_{1i}\tau_{i+1} + E_{1i}\tau_i + F_{1i}\tau_{i-1} = G_{i1}.$$

* p_s is the pressure in the vicinity of a singular point

Symbol i is the index of the point on the predetermined series $\xi = \text{const}$ of such a point that $i = 0$ corresponds to $\bar{v}_x = 0$, $i = 1$, $\bar{v}_x = 1$.

Solving the system with respect to H_i and τ_i on a new line, using the conditions $\bar{v}_x = 0$ and $\bar{v}_x = 1$, we get the values of H'_i and $\bar{\tau}_i$ which are then used for determining these values on the next lines.

Calculations are made in two opposite directions: one, starting with the body boundary with fixed conditions at $\bar{v}_x = 0$ and the second, vice versa, from the boundary layer edge with fixed conditions at $\bar{v}_x = 1$. The conditions on the singular line $\xi = 0$ are obtained from equation (4.6). Assuming the existence and the only possible way of solving system, equation (4.7), with boundary conditions, equation (4.8), the solution being regular in the region $0 < \bar{v}_x < 1$ to $\epsilon < \xi < \infty$ one may come to the conclusion that:

$$\frac{\partial H'}{\partial \xi} = \frac{\partial \bar{\tau}}{\partial \xi} = 0 \text{ at } \xi = 0.$$

Because of the use of power dependences for air properties in [35] finite limits of coefficients at $\partial H'/\partial \xi$ and at $\partial \tau/\partial \xi$ in equation (4.7) were found.

Thus, to determine H', $\bar{\tau}$ on $\xi = 0$ we get a system of ordinary differential equations with two point boundary conditions, the solution of which is found by the Runge–Kudt–Hill method. The method mentioned above was used for calculating both a perfect dissociating gas and pressure distribution according to the Newton modified theory. $(p_0/p_s) = Sm^2\theta$.

Difference approximation, equation (4.8), may be replaced by the relations establishing the connexion between the values of the function and those of its derivatives at three adjacent points:

$$\xi = \xi_i; \quad \xi = \xi_{i+1} = \xi_i + h;$$
$$\xi = \xi_{i+2} = \xi_i + 2h \ldots;$$

where h is the step by ξ.

$$F(\xi_{i+2}, \bar{v}_x) = F(\xi_i \bar{v}_x) + 2h \left.\frac{\partial F}{\partial \xi}\right|_{\xi=\xi_{i+1}} + 0(h^3); \qquad (4.10)$$

$$\frac{\partial^n F(\xi_{i+2}, \bar{v}_x)}{\partial \bar{v}_x^n} = \frac{\partial^n F(\xi_i, \bar{v}_x)}{\partial \bar{v}_x^n} + 2h \left.\frac{\partial^{n+1} F(\xi, \bar{v}_x)}{\partial \xi \partial \bar{v}_x^n}\right|_{\xi=\xi_{i+1}} + 0(h^3); \qquad (4.11)$$

$$\left.\frac{\partial F}{\partial \xi}\right|_{\xi-\xi_{i+1}} = \frac{2}{h} F(\xi_{i+2}, \bar{v}_x) - \left[\frac{4}{h} F(\xi_{i+1}, \bar{v}_x) - \frac{2}{h} F(\xi_i \bar{v}_x) - \left.\frac{\partial F(\xi_1 \bar{v}_x)}{\partial \xi}\right|_{\xi=\xi_i}\right] + 0(h^2).$$

We understand the function F to be one of the functions H' or $\bar{\tau}$.

From equations (4.7), with the help of equation (4.11), we may eliminate the derivatives of H' and $\bar{\tau}$ with respect to ξ at $\xi = \xi_{i+2}$; and with the help of equation (4.10) in the coefficients A_2, $B_2 \ldots$ we may substitute all the values at $\xi_{i+2} = \xi$ for the values at the points $\xi = \xi_i$ and $\xi = \xi_{i+1}$. Thus, for the line $\xi = \xi_{i+2}$ we get a system of ordinary differential equations differing from exact system of equation (4.7) in values of the order h^2.

The derivatives

$$\frac{\partial H'}{\partial \xi}, \left.\frac{\partial \tau}{\partial \xi}\right|_{\xi=\xi_{i+2}},$$

necessary for determining the approximate values of the functions, and their first derivatives at $\xi = \xi_{i+3}$ are determined from equation (4.7). Boundary conditions are linearized in an analogous way.

It is clear that for the boundary layer calculation by successive transition from the straight line $\xi = \xi_i$ to that $\xi = \xi_{i+1}$ while increasing ξ, one must know the values:

$$F(0, \bar{v}_x), \quad \frac{\partial F(0, \bar{v}_x)}{\partial \bar{v}_x}, \quad \left.\frac{\partial F}{\partial \xi}\right|_{\xi=\xi_1=h} \quad \left.\frac{\partial^2 F}{\partial \bar{v}_x^2}\right|_{\xi=\xi_1=h},$$

i.e. it is necessary to solve equation (4.7) at the points $\xi_0 = 0$, $\xi_1 = h$ in the neighbourhood of a front edge. At sufficiently small ξ, such a solution may be found with the help of expansion of H' and $\bar{\tau}$ in powers of ξ in the neighbourhood of the front edge:

$$H' = H'_{(0)}(\bar{v}_x) + H'_{(1)}(\bar{v}_x)\xi^{1/2} + H'_{(2)}(\bar{v}_x)\,\xi + \ldots$$
$$\bar{\tau} = \bar{\tau}_{(0)}(\bar{v}_x) + \bar{\tau}_{(1)}(\bar{v}_x)\,\xi^{1/2} + \bar{\tau}_{(2)}(\bar{v}_x)\,\xi + \ldots \qquad (4.12)$$

Substituting the series of equation (4.12) into equation (4.7) and equating coefficients at equal powers of ξ to zero, we shall obtain a sequence of systems of ordinary differential equations for determining coefficients of series:

$$H'_{(0)},\quad H_{(1)},\quad \bar{\tau}_{(0)},\quad \bar{\tau}_{(1)},\ldots$$

The equations for $H'^{(0)}$ and $\bar{\tau}^{(0)}$ are non-linear, while the rest are linear ones. The derivatives of the functions with respect to ξ at $\xi = h$ are determined from equation (4.7) on the basis of the known values of the functions and their derivatives with respect to $\bar{v}_x$ at the point $\xi_1 = h$, calculated with the help of the series in equation (4.12).

It is true that in this case there is disagreement in the values of the functions and their derivatives with respect to $\bar{v}_x$ calculated from equation (4.12), near the surface $\bar{v}_x = 0$; therefore direct determination of the derivatives by ξ at $\xi_1 = h$ and $v_x = 0$ proves to be impossible. The values of the functions and their derivatives can be corrected near $\xi = 0$ using the series, equation (4.12), the approximations, equation (4.11), and the system, equation (4.7). This method, as well as all the methods described in section 4, may be applied to more complicated problems.

This method was applied by Kulonen [36] to solving the problem of the laminar boundary layer of a wing in the presence of injection.

(6) *Application of small disturbance method to laminar boundary layer problems*

If the velocity outside the boundary layer and the air properties dependent on temperature may be represented as small parameters which may be expanded with respect to $\epsilon_i(\epsilon_i^2 \sim 0)$, then it is possible in many cases to solve laminar boundary layer problems by the method of small disturbances. The essence of the method lies in disturbance of the known solutions of boundary layer equations, expansion being done by the definite parameter.

This method has been applied to laminar boundary layer problems in the case of thermodynamic equilibrium, with an impenetrable wall and constant temperature distribution along the surface [37]. This method may also be applied to more complicated cases.

5. TURBULENT BOUNDARY LAYER

When solving turbulent boundary layer problems for a gas mixture one may generalize the semi-empirical theory of turbulence. Such a generalization is given in our work [1]. In it there is a complete solution of the problem on the turbulent boundary layer of a plate assuming the equality of mixing lengths $l = l_d = l_h = Ky$, where y is the distance from the wall.

This allows one to write integrals of energy and diffusion equations for the turbulent layer:

$$H = av_x + b,\quad \xi_i = a_1 v_x + b_1.$$

In the laminar sub-layer at Le_i and Pr numbers not equal to unity, H is assumed to be represented as the polynomial of the second power of v_x. When solving the problem, friction stress is taken to be:

$$\tau = \tau_w\left[1 - \left(\frac{y}{\delta}\right)^2\right] + \delta\frac{\mathrm{d}p}{\mathrm{d}x}\left[\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right],$$

where it is assumed that $T/M = c_1 h + d$, and, as usual, integral relations are used.

For determining the heat flow and the friction coefficients the following formulae were obtained in the work:

$$q_w = \frac{\lambda_w}{c_p}\frac{\tau_w}{\mu_w}\left\{Pr^{1/3}\frac{H_\infty - H_w - Aa(Pr)\,(u_l^2/2)}{u[1 - a(Pr)\,(u_l/u)]} - Pr\frac{f_1\,(u_l^2/2) - Pr\,f_2 u_l}{u[1 - a(Pr)\,(u_l/u)]} - Pr\,f_2\right\},$$

$$\tau_w = \tfrac{1}{2}c_f\rho_0 u^2,\quad c_f = \frac{1 - \bar{u}^2}{\bar{H}_w}\,\frac{2}{\xi^2}$$

$$n_1 + n_2\frac{c}{\bar{u}_w}K\xi = \ln\left\{\frac{K^2}{D}\,\frac{c^2}{\bar{u}_w}\sqrt{\left(\frac{1 - \bar{u}^2}{\bar{H}_w}\right)}\,\frac{ux}{\nu_w}\right\},$$

where

$$a(Pr) = 1 - \sqrt{Pr};\quad u_l = \frac{K_1}{K}\sqrt{\left(\frac{\tau_w}{\rho_w}\right)};$$

$$f_1 = Pr\,\mu_w\frac{\left\{\frac{\partial}{\partial y} - \frac{\mu}{Pr}\sum_i\left[(Le_i - 1)h_i\frac{\partial\xi_i}{\partial y}\right]\right\}_{y=0}}{\tau_w^2};$$

$$f_2 = f_1\frac{u_l}{Pr} + \frac{\lambda}{c_p\tau_w}\left(\sum_i(Le_i - 1)h_i\frac{\partial\xi_i}{\partial y}\right)_{y=\delta l}$$

$$\frac{K_1}{K} \simeq 11;\quad K \approx 0{\cdot}4;\quad n_1 \simeq 2{\cdot}9;\quad n_2 \approx 1{\cdot}16;$$

$$c = \arc\sin \frac{2\bar{u}^2 + \bar{H}_w - 1}{\beta} - \arc\sin \frac{\bar{H}_w - 1}{\beta};$$

$$D = \frac{K_1}{2K^2} \exp \left\{1 - K_1 - K_1 \frac{a_1}{\beta^2} \{(\bar{H}_w + 1) - 2\sqrt{[(1 - \bar{u}^2)\bar{H}_w]}\}\right\};$$

$$a_1 = (1 - \bar{H}_w)(1 - Pr) + \frac{Pr f_2}{u} \bar{u}^2;$$

$$\beta^2 = 4\bar{u}^2\bar{H}_w + (\bar{H}_w - 1)^2;$$

$$\bar{u} = \frac{u}{\sqrt{\{2[H_\infty + (d/c_1)]/A\}}};$$

$$\bar{u}_w = \frac{u}{\sqrt{\{2[H_w + (d/c)]/A\}}};$$

$$\bar{H}_w = \frac{H_w + (d/c)}{H_\infty + (d/c_1)}.$$

From the formulae obtained, in particular cases, the known results are obtained for an incompressible liquid. In the case of a thermo-isolated wing or an axisymmetrical body, where $Pr = Le_i = 1$, the problem is solved simply. Here, it may be assumed that $H = \text{const}$, which allows one to obtain expressions for the friction coefficient and other values, which generalize the solutions known.

For the non-thermo-isolated wing or the axisymmetrical body, the problem of gas motion in a turbulent boundary layer has not been solved.

Lately some works devoted to the consideration of turbulent boundary layer, generalizing the analogy method have been published; it is supposed in these works that the power velocity distribution law is valid in the boundary layer $(v_x/u) = (\eta/\delta)^n$. Of these, Belyanin's work, which was reported at the Congress of Mechanic Workers in January 1960, should be mentioned.

REFERENCES

1. I. P. GINZBURG, *Vestnik Leningradskogo Gosudarstvennogo Universiteta*, N 1 (1961).
2. V. S. AVDUEVSKY and E. I. OBROSKOVA, *Izv. AN S.S.R. Mekhanika i Mashinostroenie*, N 4 (1960).
3. E. C. STUPACHENKO, B. S. DOTSENKO, I. M. STAKHANOV and E. V. SAMUILOV, Fizicheskaya gazodinamika, *Sbornik Izd. AN SSSR, Moskva* (1959).
4. S. CHAPMAN and T. COWLING, *The mathematical theory of non-uniform gases*. Cambridge (1939).
5. R. FAULER and E. GUGENGEIM, *Statisticheskaya termodinamika*. Gosinizdat, Moskva (1949).
6. A. I. GUBANOV and Y. P. LUNKIN, *Zh. Tekh. Fiz.* XXVI, N 11 (1957).
7. M. BODENSTEN, *Roczinike Chemie*, **18**, 374 (1938).
8. G. CARERI, *Nuovo Cimento*, **7**, 155 (1950).
9. A. S. PREDVODITELEV, E. W. STUPACHENKO, J. P. STAKHANOV, E. V. SAMUILOV, A. S. PLESHANOV and J. B. ROGDESTVENSKY, Tablitsa termodinamichekikh funktsii vozdukha v intervale temperatur ot 1000 do 12000° v intervale davlenii ot 0·001 do 1000 atm, *Izd-vo AN SSSR* (1957).
10. MOORE, Reshenie uravnenii laminarnogo pogranichnogo sloya s uchetom dissotsiatsii dlya szhimaemoi zhidkosti s izmenyayuscchimisya fizicheskimi svoistvami, *Mekhanika, Sbornik perevodov*, N 5 (1953).
11. E. P. VAULIN, *Dokl. AN SSSR*, **113**, N 6 (1957).
12. C. COHEN, R. BROMBERG and R. LIPKIS, *VRT*, N 4 (1959).
13. R. E. DORFMAN, N. J. POLSKY and S. W. ROMANENKO, *PMM*, vypusk 2, 22 (1958).
14. S. M. SCALA and C. W. BAULKNIGHT, *VRT*, N 12 (1960).
15. S. LEVY, *JAS*, **21**, N 7 (1954).
16. T. J. LI and H. T. NAGAMATSI, *JAS*, **22**, 607–616 (1955).
17. C. COHEN and E. RESHOTKO, *NACA*, *Rep*. 1294 (1956).
18. E. RESHOTKO and C. COHEN, *NACA*, *Tech. Note*, 3513 (1953); 3525 (1955).
19. L. LEES, *Jet Propulsion*, **26**, N 4 (1956).
20. J. A. FAY and F. R. RIDDELL, *JAS*, **25**, N 2 (1958).
21. J. A. FAY, F. R. RIDDELL and N. H. KEMP, *Jet Propulsion*, **27** (1957).
22. KALIKHMAN, *Trudy* NII, N 1 (1947).
23. A. G. HEMMIT, Vliyanie vyazkosti na giperzvukovoe obtekanie ploskoi plastiny s konechnoi perednei kromkoi, *Mekhanika, Sbornik perevodov*, N 5 (1959).
24. N. A. TOLMACHEV, *Vestnik MGU*, v. 172, *Mekhanika*, 5 (1954).
25. B. THWAITES, *Aero. Quart.*, I, 245–280 (1949).
26. N. ROTT and L. F. CRABTREC, *J. Aero. Sci.*, **19**, 553–565 (1952).
27. W. HAYES, Voprosy raketnoi tekhniki, *Sbornik perevodov*, N 3 (1957).
28. N. E. KOCHIN and L. G. LOITSYANSKY, *DAN SSSR*, XXXVI, N 9 (1942).
29. V. S. AVDUEVSKY and R. M. KOPYATKEVICH, *Izv. AN SSSR, OTN, Mekhanika i mashinostroenie*, N 1 (1960).
30. H. GÖRTLER, *J. Math. Mech.*, **8**, 1–66 (1957).
31. E. M. SPARROW, *J. Fluid Mech.*, **4**, 321–392 (1958).
32. M. E. SHVETS, O priblizhennom reshenii nekotorykh zadach gidrodinamiki pogranichnogo sloya, *PMM*, 13, vy. p. 3 (1949).
33. Z. S. GALANOVA, *Vestnik LGU*, N 19 (1960).
34. G. A. KULONEN, *Vestnik LGU*, N 19 (1960).
35. R. T. KRAMER and H. M. LIBERSTEIN, *JAS*, N 8 (1959).
36. G. A. KULONEN, *Vestnik LGU*, N 1 (1961).
37. V. V. LUNEV, Primenenie metoda malykh vozmushchenii k zadacham laminarnogo pogranichnogo sloya, *PMM*, vyp. 5, XXI, (1957).
38. C. R. WILKE, *J. Chem. Phys.* **18**, No. 4 (1950).

Abstract—The paper present equations for a gas mixture flow in laminar and turbulent boundary layers with a discussion of possible methods of their solution.

In particular, some new classes of similar solutions and a scheme for an approximate solution for the equations with the use of the Kochin–Loitsyansky method are mentioned. A series of new results on a turbulent boundary layer is given.

Résumé—L'article donne des équations relatives à un mélange de gaz en écoulement dans les couches limites laminaire et turbulente et discute des méthods possibles pour les résoudre.

En particulier, des classes nouvelles de solutions semblables et le schéma d'une solution approchée utilisant la méthode de Kochin–Loitsyansky y sont mentionnés. Une série de nouveaux résultats sur la couche limite turbulente est donnée.

Zusammenfassung—Für die laminare und turbulente Grenzschichtströmung eines Gasgemisches werden Gleichungen mit möglichen Lösungsmethoden angegeben.

Insbesondere sind neue Arten von Ähnlichkeitslösungen erwähnt und eine Möglichkeit der näherungsweisen Lösung mit Hilfe der Kochin–Loitsyansky Methode. Für die turbulente Grenzschicht ist eine Reihe neuer Ergebnisse angegeben.